C3 June 13 (replaced paper) uu

1. Given that

$$\frac{3x^4 - 2x^3 - 5x^2 - 4}{x^2 - 4} \equiv ax^2 + bx + c + \frac{dx + e}{x^2 - 4}, \quad x \neq \pm 2$$

find the values of the constants a, b, c, d and e.

(4)

$\times$	$3x^2$	$-2x$	$+7$	Rem
x^2	$3x^4$	$-2x^3$	$+7x^2$	$-8x$
-4	$-12x^2$	$+8x$	-28	$+24$

$$3x^2 - 2x + 7 \quad \frac{-8x+24}{x^2-4}.$$

2. Given that

$$f(x) = \ln x, \quad x > 0$$

sketch on separate axes the graphs of

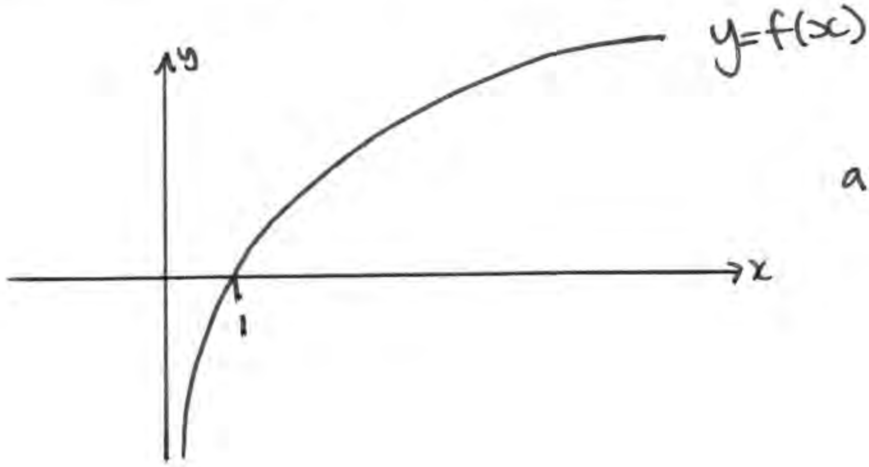
(i) $y = f(x)$,

(ii) $y = |f(x)|$,

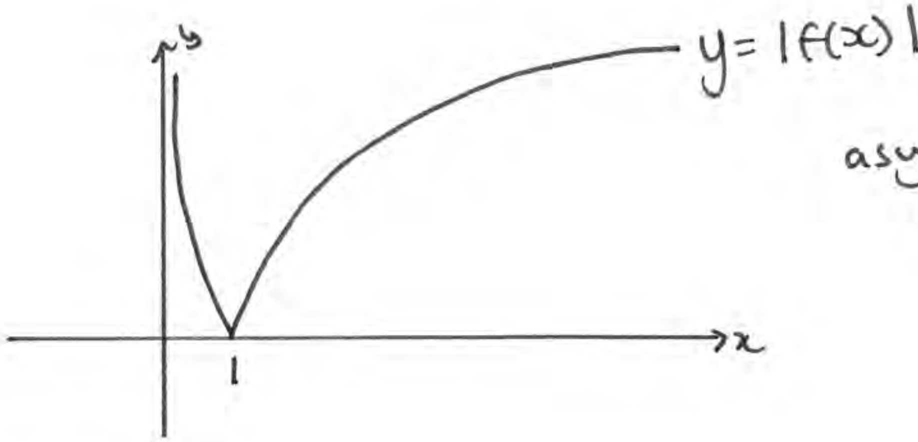
(iii) $y = -f(x - 4)$.

Show, on each diagram, the point where the graph meets or crosses the x-axis.
In each case, state the equation of the asymptote.

(7)



asymptote
$x = 0$



asymptote
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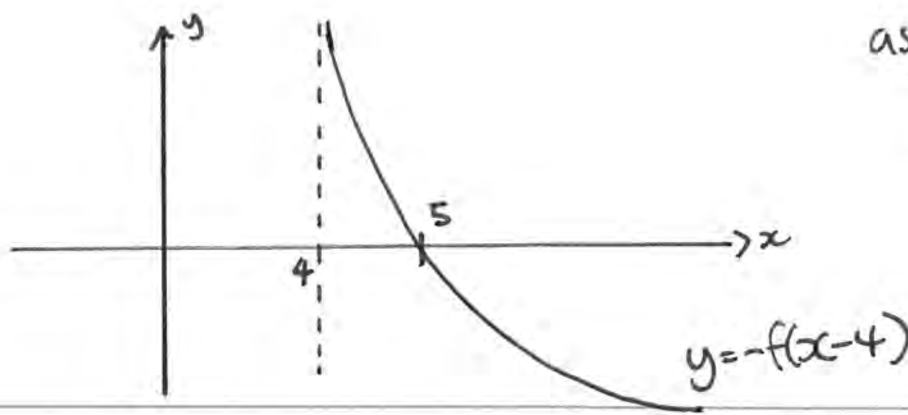


asymptote
$x = 4$

3. Given that

$$2\cos(x + 50)° = \sin(x + 40)°$$

(a) Show, without using a calculator, that

$$\tan x° = \frac{1}{3}\tan 40°$$

(4)

(b) Hence solve, for $0 \leqslant \theta < 360$,

$$2\cos(2\theta + 50)° = \sin(2\theta + 40)°$$

giving your answers to 1 decimal place.

(4)

$2\cos x \cos 50 - 2\sin x \sin 50 = \sin x \cos 40 + \cos x \sin 40$

$(\div \cos x)$ $\quad 2\cos 50 - 2\tan x \sin 50 = \tan x \cos 40 + \sin 40$

$(\cos 50 = \sin 40)$ $(\sin 50 = \cos 40)$

$\Rightarrow 2\sin 40 - 2\tan x \cos 40 = \tan x \cos 40 + \sin 40$

$(\div \cos 40)$

$\Rightarrow 2\tan 40 - 2\tan x = \tan x + \tan 40$

$\Rightarrow \tan 40 = 3\tan x \quad \therefore \tan x = \frac{1}{3}\tan 40$

#

b) $\tan x \Rightarrow \tan 2\theta = 0.2796\ldots$

$2\theta = 15.63, 195.63, 375.63, 555.63$

$\div 2 \Rightarrow \theta = \underline{7.8}, \underline{97.8}, \underline{187.8}, \underline{277.8}$

4. $$f(x) = 25x^2e^{2x} - 16, \qquad x \in \mathbb{R}$$

(a) Using calculus, find the exact coordinates of the turning points on the curve with equation $y = f(x)$.

(5)

(b) Show that the equation $f(x) = 0$ can be written as $x = \pm\frac{4}{5}e^{-x}$

(1)

The equation $f(x) = 0$ has a root α, where $\alpha = 0.5$ to 1 decimal place.

(c) Starting with $x_0 = 0.5$, use the iteration formula

$$x_{n+1} = \frac{4}{5}e^{-x_n}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

(d) Give an accurate estimate for α to 2 decimal places, and justify your answer.

(2)

a) $u = 25x^2 \quad v = e^{2x} \quad \Rightarrow f'(x) = 50xe^{2x} + 50x^2e^{2x}$
$u' = 50x \quad v' = 2e^{2x}$

at TP $f'(x) = 0 \Rightarrow 50e^{2x}(x + x^2) = 0$
$\Rightarrow 50xe^{2x}(1 + x) = 0$

$50xe^{2x} = 0$ if $x = 0 \Rightarrow y = -16$
$(1+x) = 0$ if $x = -1 \Rightarrow y = 25e^{-2} - 16$

$(0, -16)$; $(-1, 25e^{-2} - 16)$

b) $25x^2e^{2x} - 16 = 0 \Rightarrow 25x^2e^{2x} = 16$

$\Rightarrow 25x^2 = \frac{16}{(e^x)^2}$ $(\sqrt{\ })$ $\Rightarrow 5x = \pm\frac{4}{e^x}$

$\Rightarrow 5x = \pm 4e^{-x} \Rightarrow x = \pm\frac{4}{5}e^{-x}$ #

c) $x_0 = 0.5$
$x_1 = 0.485$
$x_2 = 0.492$
$x_3 = 0.489$

$\rightarrow x_n = 0.49\ldots$

d) $f(0.485) = -0.49 < 0$
$f(0.495) = +0.49 > 0$

$\therefore$ by sign change rule
$\alpha = 0.49$ (2dp).

5. Given that

$$x = \sec^2 3y, \qquad 0 < y < \frac{\pi}{6}$$

(a) find $\dfrac{dx}{dy}$ in terms of y.

(2)

(b) Hence show that

$$\frac{dy}{dx} = \frac{1}{6x(x-1)^{\frac{1}{2}}}$$

(4)

(c) Find an expression for $\dfrac{d^2y}{dx^2}$ in terms of x. Give your answer in its simplest form.

(4)

a) $x = (\sec 3y)^2$

$\Rightarrow \dfrac{dx}{dy} = 2(\sec 3y)^1 \times 3\sec 3y \tan 3y$

$\Rightarrow \dfrac{dx}{dy} = 6\sec^2 3y \tan 3y$

b) $\dfrac{dy}{dx} = \dfrac{1}{6\sec^2 3y \tan 3y}$

$x = \sec^2 3y$

$x = \tan^2 3y + 1$

$x - 1 = \tan^2 3y$

$\sqrt{x-1} = \tan 3y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{6x\sqrt{x-1}}$ #

$$\frac{dy}{dx} = \frac{x^{-1}}{6(x-1)^{\frac{1}{2}}} \qquad u = x^{-1} \qquad V = (x-1)^{\frac{1}{2}} \times 6$$

$$u' = -1x^{-2} \qquad V' = 3(x-1)^{-\frac{1}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-\frac{6(x-1)^{\frac{1}{2}}}{x^2} - \frac{3}{x(x-1)^{\frac{1}{2}}}}{36(x-1)}$$

$$\Rightarrow \frac{\frac{-6(x-1)^{\frac{1}{2}}(x-1)^{\frac{1}{2}} - 3x}{x^2(x-1)^{\frac{1}{2}}}}{36(x-1)}$$

$$\Rightarrow \frac{-6(x-1)-3x}{36x^2(x-1)^{3/2}} = \frac{6-9x}{36x^2(x-1)^{3/2}}$$

$$\Rightarrow \frac{2-3x}{12x^2(x-1)^{3/2}}$$

#

6. Find algebraically the exact solutions to the equations

(a) $\ln(4-2x)+\ln(9-3x)=2\ln(x+1), \qquad -1<x<2$ **(5)**

(b) $2^x e^{3x+1}=10$

Give your answer to (b) in the form $\dfrac{a+\ln b}{c+\ln d}$ where a, b, c and d are integers. **(5)**

a) $\cancel{\ln}[(4-2x)(9-3x)]=\cancel{\ln}[(x+1)^2]$

$\Rightarrow 6x^2-30x+36=x^2+2x+1$

$\Rightarrow 5x^2-32x+35=0 \Rightarrow (5x-7)(x-5)=0$

$\therefore x=\frac{7}{5} \quad x=5$

b) $\ln(2^x e^{3x+1})=\ln 10$

$\Rightarrow \ln 2^x+\ln e^{3x+1}=\ln 10$

$\Rightarrow x\ln 2+3x+1=\ln 10$

$\Rightarrow x(3+\ln 2)=-1+\ln 10$

$\therefore x=\dfrac{-1+\ln 10}{3+\ln 2}$

7. The function f has domain $-2 \leqslant x \leqslant 6$ and is linear from $(-2, 10)$ to $(2, 0)$ and from $(2, 0)$ to $(6, 4)$. A sketch of the graph of $y = \mathrm{f}(x)$ is shown in Figure 1.

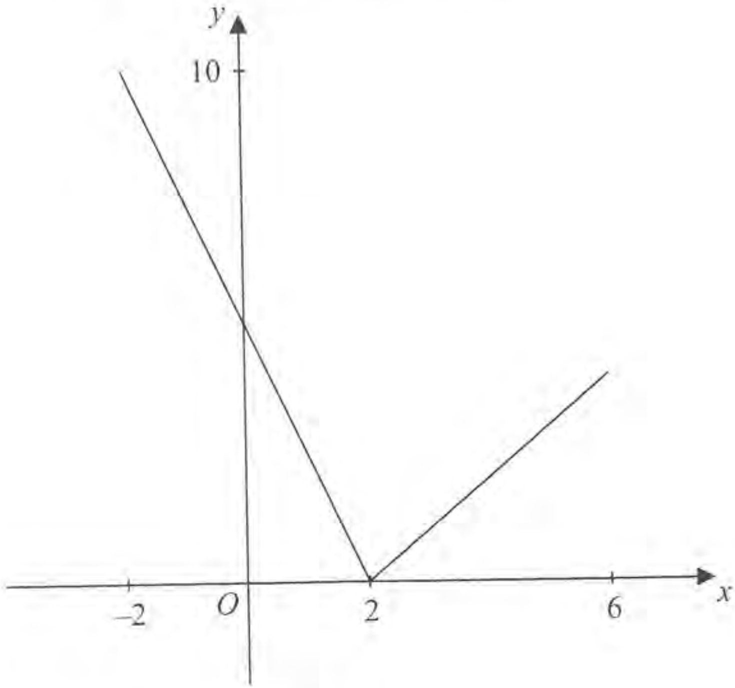


Figure 1

(a) Write down the range of f.

(1)

(b) Find ff(0).

(2)

The function g is defined by

$$\mathrm{g} : x \to \frac{4 + 3x}{5 - x}, \qquad x \in \mathbb{R}, \quad x \neq 5$$

(c) Find $\mathrm{g}^{-1}(x)$

(3)

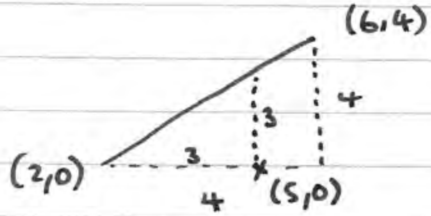
(d) Solve the equation $\mathrm{gf}(x) = 16$

(5)

a) $0 \leq y \leq 10$

b) $f(0) = 5$

$ff(0) = f(5) = 3$



c) $x = \dfrac{4+3y}{5-y} \Rightarrow 5x - xy = 4 + 3y$

$\Rightarrow 3y + xy = 5x - 4 \Rightarrow y(3+x) = 5x - 4$

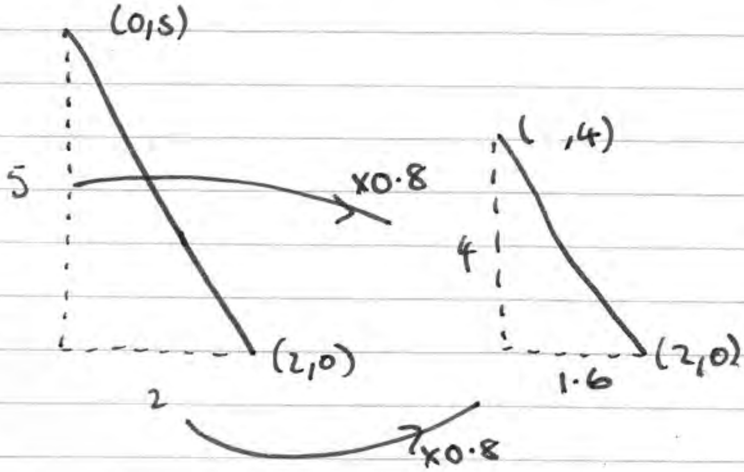
$\therefore y = g^{-1}(x) = \dfrac{5x-4}{3+x}$

d) $gf(x) = 16 \Rightarrow \dfrac{4+3f(x)}{5-f(x)} = 16$

$\Rightarrow 4 + 3f(x) = 80 - 16f(x)$

$\Rightarrow 19f(x) = 76 \Rightarrow f(x) = 4 \Rightarrow \underline{x = 6.}$

ii)



$\therefore \underline{x = 0.4}$

8.

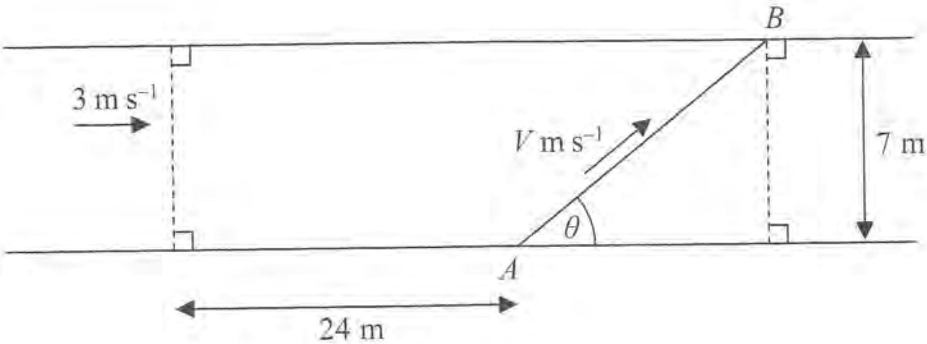


Figure 2

Kate crosses a road, of constant width 7 m, in order to take a photograph of a marathon runner, John, approaching at 3 m s^{-1}.
Kate is 24 m ahead of John when she starts to cross the road from the fixed point A.
John passes her as she reaches the other side of the road at a variable point B, as shown in Figure 2.
Kate's speed is $V \text{ m s}^{-1}$ and she moves in a straight line, which makes an angle θ, $0 < \theta < 150°$, with the edge of the road, as shown in Figure 2.

You may assume that V is given by the formula

$$V = \frac{21}{24\sin\theta + 7\cos\theta}, \qquad 0 < \theta < 150°$$

(a) Express $24\sin\theta + 7\cos\theta$ in the form $R\cos(\theta - \alpha)$, where R and α are constants and where $R > 0$ and $0 < \alpha < 90°$, giving the value of α to 2 decimal places.

(3)

Given that θ varies,

(b) find the minimum value of V.

(2)

Given that Kate's speed has the value found in part (b),

(c) find the distance AB.

(3)

Given instead that Kate's speed is 1.68 m s^{-1},

(d) find the two possible values of the angle θ, given that $0 < \theta < 150°$.

(6)

a) $R\cos(\theta-\alpha) = R\cos\cancel{\theta}\cos\alpha + R\sin\cancel{\theta}\sin\alpha$

$\qquad\qquad\qquad\qquad 7\cos\cancel{\theta} + 24\sin\cancel{\theta}$

$\therefore \dfrac{R\sin\alpha = 24}{R\cos\alpha = 7} \Rightarrow \tan\alpha = \dfrac{24}{7} \Rightarrow \alpha = 73.74$

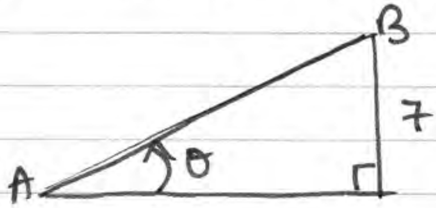
$R = 25$

$= 25\cos(\theta - 73.74)$

b) $V_{min} = \dfrac{21}{\max(25\cos(\theta-73.74)} = \dfrac{21}{25}$

c) $25\cos(\theta-73.74)$ is max when

$\theta - 73.74 = 0 \Rightarrow \theta = 73.74.$



$\sin 73.74 = \dfrac{7}{AB}$

$\therefore AB = \dfrac{175}{24} = 7.29$

d) $\dfrac{21}{25\cos(\theta-73.74)} = 1.68$

$\Rightarrow \cos(\theta-73.74) = \dfrac{21}{25\times1.68} = 0.5$

$\therefore \theta - 73.74 = 60, 300, -60, \ldots$

$+73.74$

$\therefore \theta = 13.74, 133.74$